Lattice supersymmetry: equivalence between the link approach and orbifolding

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ABSTRACT: We examine the relation between supersymmetric lattice gauge theories constructed by the link approach and by orbifolding and show that they are equivalent. We discuss the number of preserved supersymmetries.

Contents

1. Introduction

A number of apparently different ways to preserve exactly both gauge symmetry and some supersymmetries on a euclidean space-time lattice have been proposed. One is based on the so-called orbifolding method $[1-5]$, in which dimensional reduction to a zero-dimensional mother theory is followed by an orbifold projection that leaves invariant both a fixed number of supersymmetries and a discrete symmetry. The latter can be viewed as translations on a space-time lattice that appears as a result of "deconstruction". Two apparently different formulations, pursued by Catterall $[6-8]$ and Sugino $[9-11]$, use as starting points ideas from topological field theory in order to preserve exactly a number of supersymmetries on a space-time lattice. Recently it has been demonstrated how both of these formulations can be understood from the point of view of orbifolding and deconstruction as well [13, 14]. Finally, an approach that is again tied up closely with both topological field theory (twisted supersymmetry) and the Dirac-Kähler formulation of lattice fermions has been advocated $[15-17]$. We will call this latter approach to lattice supersymmetry for the *link approach*. What is unique to that formulation is the claim that it can preserve exactly *all* supersymmetries at finite lattice spacings, not just those associated with the nilpotent charges related to the underlying topological field theories. There has recently been some discussion about this issue [18, 19].

Because also the result of the link approach resembles so much that of orbifolding, one would like to understand better the relationship between the two formalisms. In this paper we show that the link approach is completely equivalent to the one based orbifolding. We limit ourselves to describing this equivalence in detail for the case of two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric Yang-Mills theory. It will be clear from our discussion that the equivalence trivially generalizes. This equivalence between the two formulations makes it more urgent to understand also the number of preserved supersymmetries on the lattice. This prompts us to investigate the fate of those supersymmetries that are lost in the orbifolding procedure, and only hoped to be regained in the continuum limit. As we shall show, the additional supersymmetry transformations of the link approach have a natural

explanation in terms of the orbifolding procedure. As expected, they correspond to field transformations that violate the Leibniz rule of field variations, and we thus cannot see these additional transformations as symmetries of the action. Nevertheless, the origin of these transformations can be clearly understood from the orbifolding point of view. In this way, the apparent discrepancy in terms of the number of preserved supersymmetries in the two formulations is resolved.

The organization of this paper is as follows. In the next section, we briefly review the orbifold projection of the zero-dimensional Yang-Mills matrix theory (the mother theory), and explain how several of the supersymmetries are broken by the orbifold projection. We compare the action with that of the link approach, and show how the most general orbifolded action (which does not preserve any supercharges at all) is in one-to-one correspondence with that of the link approach. The shift parameters of the link approach are identified with the U(1) charges of the supersymmetry generators in the orbifolded action. In section 3, we investigate the fate of the broken supersymmetries. We show that the would-be transformations agree exactly with those of the link approach if we allow for a redefinition of the fermionic parameters, and we discuss the interpretation of these supersymmetry transformations. Section 4 contains our conclusions.

2. Supersymmetry transformations in the mother theory

We begin by briefly recalling the main ingredients in the orbifold construction of super-symmetric lattice gauge theories.[1] Because the points we shall focus on are not specific in regards to, for example, dimensionality, we restrict ourselves to the $\mathcal{N} = (2,2)$ super-symmetric gauge theory in two space-time dimensions. The action of the mother theory is in this case obtained by dimensional reduction of four-dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills theory:

$$S_m = \frac{1}{g^2} \mathrm{Tr} \left(\frac{1}{4} v_{\alpha\beta}^2 + \bar{\psi} \bar{\sigma}_\alpha [v_\alpha, \psi] \right), \tag{2.1}$$

where $\alpha, \beta = 0, \cdots, 3$, v_α are Hermitian bosonic matrices, ψ, $\bar{\psi}$ are independent two-components spinors, and $v_{\alpha\beta} = i[v_\alpha, v_\beta]$. For the purpose for the future discussion, we assume the gauge group of the theory to be U(kN^2). In the following, we use the notation,

$$\sigma_\alpha = (1_2, -i\tau_i), \qquad \bar{\sigma}_\alpha = (1_2, i\tau_i), \tag{2.2}$$

where τ_i ($i = 1, 2, 3$) are the Pauli matrices. In addition to the gauge symmetry, $v_\alpha \rightarrow g^{-1} v_\alpha g, \cdots$, this theory is invariant under the "global" symmetry SO(4) × U(1), which corresponds to the Lorentz symmetry and the R-symmetry of the four-dimensional $\mathcal{N} = 1$ SYM theory [2], respectively. Furthermore, the action (2.1) is invariant under the following supersymmetry transformation:

$$\begin{aligned}
\delta v_\alpha &= -i\bar{\psi} \bar{\sigma}_\alpha \xi + i\bar{\xi} \bar{\sigma}_\alpha \psi, \\
\delta \psi &= -i v_{\alpha\beta} \sigma_{\alpha\beta} \xi, \\
\delta \bar{\psi} &= i v_{\alpha\beta} \bar{\xi} \bar{\sigma}_{\alpha\beta},
\end{aligned} \tag{2.3}$$

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[1]For a nice review, see, e.g., ref. [20].

	z_1	z_2	η	χ_{12}	ψ_1	ψ_2
q_1	1	0	1/2	-1/2	1/2	-1/2
q_2	0	1	1/2	-1/2	-1/2	1/2
q_3	0	0	1/2	1/2	-1/2	-1/2

Table 1: The charge assignment of the maximal U(1) symmetries

where ξ and $\bar{\xi}$ are constant Grassmann-odd spinor parameters.

Following ref. [2], we define complex fields z_m and $\bar{z}_m$ $(m = 1, 2)$ by

$$
\begin{aligned}
z_1 &= -iv_1 + v_2, & \bar{z}_1 &= iv_1 + v_2, \\
z_2 &= v_0 + iv_3, & \bar{z}_2 &= v_0 - iv_3,
\end{aligned}
\tag{2.4}
$$

and express the component fields of ψ and $\bar{\psi}$ as

$$
\psi = \begin{pmatrix} \chi_{12} \\ \eta \end{pmatrix}, \qquad \bar{\psi} = (\psi_1, \psi_2).
\tag{2.5}
$$

Using these fields, the action of the mother theory (2.1) can be rewritten as

$$
S_m = \frac{1}{g^2}\text{Tr}\left(\frac{1}{4}|[z_m, z_n]|^2 + \frac{1}{8}[z_m, \bar{z}_m]^2 + \eta[\bar{z}_m, \psi_m] - \chi_{mn}[z_m, \psi_n] \right).
\tag{2.6}
$$

In this expression, the global U(1) symmetries are manifest. In fact, one can easily show that there are three independent U(1) symmetries for which all the fields (2.4) and (2.5) have definite charges q_a $(a = 1, 2, 3)$ as shown in table 1. In terms of the new variables, the supersymmetry transformations (2.3) take the forms

$$
\begin{aligned}
\delta z_m &= 2i\hat{\kappa}\psi_m + 2i\hat{\kappa}_m\eta, \\
\delta \bar{z}_m &= -2i\hat{\kappa}_{mn}\psi_n - 2i\hat{\kappa}_n\chi_{mn}, \\
\delta \eta &= \frac{i}{2}\hat{\kappa}[z_m, \bar{z}_m] + \frac{i}{2}\hat{\kappa}_{mn}[z_m, z_n], \\
\delta \chi_{12} &= -i\hat{\kappa}[\bar{z}_1, \bar{z}_2] - \frac{i}{2}\hat{\kappa}_{12}[z_m, \bar{z}_m], \\
\delta \psi_m &= i\hat{\kappa}_n\left([z_m, \bar{z}_n] - \frac{1}{2}\delta_{mn}[z_l, \bar{z}_l]\right),
\end{aligned}
\tag{2.7}
$$

where we have expressed the components of ξ and $\bar{\xi}$ as

$$
\xi = \begin{pmatrix} \hat{\kappa}_{12} \\ \hat{\kappa} \end{pmatrix}, \qquad \bar{\xi} = (\hat{\kappa}_1, \hat{\kappa}_2),
\tag{2.8}
$$

with $\hat{\kappa}_{mn} = -\hat{\kappa}_{nm}$. We emphasize here that the transformations (2.7) correspond to a symmetry of the action if and only if the supersymmetry parameters $\hat{\kappa}$, $\hat{\kappa}_m$ and $\hat{\kappa}_{12}$ transform as singlets under the gauge group. This elementary fact, which is also obvious from the transformation law (2.3), is crucial for the discussion of the number of preserved

supersymmetries below. We can define the operators of supercharges $\{\widehat{Q}, \widehat{Q}_m, \widehat{Q}_{12}\}$ through the transformation (2.7) as

$$\delta\Phi = 2i\hat{\kappa}\widehat{Q}\Phi - 2i\hat{\kappa}_{12}\widehat{Q}_{12}\Phi + 2i\hat{\kappa}_m\widehat{Q}_m\Phi, \tag{2.9}$$

where Φ is a generic field of the theory. It is straightforward to show that the supercharges satisfy the algebra,

$$\{\widehat{Q}, \widehat{Q}_m\} = -\frac{1}{2}[\bar{z}_m, \cdot], \quad \{\widehat{Q}_{12}, \widehat{Q}_m\} = \frac{1}{2}\epsilon_{mn}[z_n, \cdot], \tag{2.10}$$

with the other anticommutators vanishing, up to use of the equations of motion.

Next, we carry out the orbifold projection. In order to obtain a two-dimensional lattice formulation, we follow the standard procedure and mod out by $Z_N \times Z_N$ which is a subgroup of the full symmetry group of the mother theory [1–4]. In this projection, the U(1) charges of the fields play crucial roles. As mentioned above, the mother theory has three independent U(1) symmetries and any linear combination of them is also a symmetry of the theory. Following [5], we define two U(1) charges so that η has zero charges,

$$\begin{aligned}
r_1 &\equiv \ell_1^1 q_1 + \ell_1^2 q_2 - (\ell_1^1 + \ell_1^2)q_3, \\
r_2 &\equiv \ell_2^1 q_1 + \ell_2^2 q_2 - (\ell_2^1 + \ell_2^2)q_3.
\end{aligned} \tag{2.11}$$

Introducing two vectors,

$$\mathbf{e}_1 \equiv \begin{pmatrix} \ell_1^1 \\ \ell_2^1 \end{pmatrix}, \quad \mathbf{e}_2 \equiv \begin{pmatrix} \ell_1^2 \\ \ell_2^2 \end{pmatrix}, \tag{2.12}$$

the charge assignments under these U(1)'s are given in table 2. As discussed in [5], the orbifold projection can be achieved by restricting the fields corresponding to the U(1) charges according to

$$\begin{aligned}
z_m &= \sum_{\mathbf{k}\in\mathbb{Z}_N^2} z_m(\mathbf{k}) \otimes E_{\mathbf{k},\mathbf{k}+\mathbf{e}_m}, & \bar{z}_m &= \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \bar{z}_m(\mathbf{k}) \otimes E_{\mathbf{k}+\mathbf{e}_m,\mathbf{k}}, \\
\eta &= \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \eta(\mathbf{k}) \otimes E_{\mathbf{k},\mathbf{k}}, & \psi_m &= \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \psi_m(\mathbf{k}) \otimes E_{\mathbf{k},\mathbf{k}+\mathbf{e}_m}, \\
\chi_{12} &= \sum_{\mathbf{k}\in\mathbb{Z}_N^2} \chi_{12}(\mathbf{k}) \otimes E_{\mathbf{k}+\mathbf{e}_1+\mathbf{e}_2,\mathbf{k}},
\end{aligned} \tag{2.13}$$

where $E_{\mathbf{k},\mathbf{l}} \equiv E_{k_1,l_1} \otimes E_{k_2,l_2}$ with $(E_{ij})_{kl} \equiv \delta_{ik}\delta_{jl}$. As a result, we obtain the orbifolded action,

$$\begin{aligned}
S_{\text{orb}} = \frac{1}{g^2}\text{Tr}\sum_{\mathbf{k}}\Bigg(&\frac{1}{4}\Big|z_m(\mathbf{k})z_n(\mathbf{k}+\mathbf{e}_m) - z_n(\mathbf{k})z_m(\mathbf{k}+\mathbf{e}_n)\Big|^2 \\
&+ \frac{1}{8}\Big(z_m(\mathbf{k})\bar{z}_m(\mathbf{k}) - \bar{z}_m(\mathbf{k}-\mathbf{e}_m)z_m(\mathbf{k}-\mathbf{e}_m)\Big)^2 \\
&+ \eta(\mathbf{k})\Big(\bar{z}_m(\mathbf{k}-\mathbf{e}_m)\psi_m(\mathbf{k}-\mathbf{e}_m) - \psi_m(\mathbf{k})\bar{z}_m(\mathbf{k})\Big) \\
&- \frac{1}{2}\chi_{mn}(\mathbf{k})\Big(z_m(\mathbf{k})\psi_n(\mathbf{k}+\mathbf{e}_m) - \psi_n(\mathbf{k})z_m(\mathbf{k}+\mathbf{e}_n) \\
&\qquad\qquad - z_n(\mathbf{k})\psi_m(\mathbf{k}+\mathbf{e}_n) + \psi_m(\mathbf{k})z_n(\mathbf{k}+\mathbf{e}_m)\Big)\Bigg). \tag{2.14}
\end{aligned}$$

	z_1	z_2	η	χ_{12}	ψ_1	ψ_2
r	$\mathbf{e}_1$	$\mathbf{e}_2$	**0**	-$\mathbf{e}_1$-$\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{e}_2$

Table 2: Two $\mathrm{U}(1)$ charges

	z_1	z_2	η	χ_{12}	ψ_1	ψ_2
r	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{a}$	$\mathbf{a}_{12}$	$\mathbf{a}_1$	$\mathbf{a}_2$

Table 3: Three $\mathrm{U}(1)$ charges

A euclidean space-time lattice action for two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theory is obtained by deconstruction: shifting $z_m(\mathbf{k})$ and $\bar{z}_m(\mathbf{k})$ by $1/a$, where a is a fundamental lattice spacing a in the orbifolded action (2.14) [2, 5]. Another way of introducing the lattice spacing a is to regard the bosonic link variables $z_m(\mathbf{k})$ and $\bar{z}_m(\mathbf{k})$ as $\frac{1}{a}e^{iaA_m(\mathbf{k})}$ and $\frac{1}{a}e^{-iaA_m^\dagger(\mathbf{k})}$, respectively [21], where $A_m(\mathbf{k})$ are not hermitian but complex matrices. If we expand the action in a, the leading contribution clearly agrees with the action which is obtained by ordinary deconstruction. In this procedure the action (2.14) can be regarded as a lattice action for two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theory.

It is important to note that there exists a trivial generalization of the lattice formulation (2.14). In [2] it is assumed that at least one fermion component has zero $\mathrm{U}(1)$ charges in order to preserve at least one supersymmetry after orbifolding (see also [5]). However, if we do not insist on preserving any supersymmetries, we can use the three independent $\mathrm{U}(1)$ charges to obtain an orbifolded action by linearly combining the $\mathrm{U}(1)$ charges in table 1. The charge assignment in this case is summarized in table 3. Here $\mathbf{e}_m$, $\mathbf{a}$, $\mathbf{a}_{12}$ and $\mathbf{a}_m$ are three-component vectors with the relations,

$$\mathbf{a} + \mathbf{a}_m = \mathbf{e}_m, \quad \mathbf{a}_{12} + \mathbf{a}_m = -|\epsilon_{mn}|\mathbf{e}_n, \quad \mathbf{a} + \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_{12} = 0. \tag{2.15}$$

Using this notation, we obtain the following more general orbifolded action:

$$
\begin{aligned}
S_{\mathrm{orb}} = \frac{1}{g^2}\mathrm{Tr}\sum_{\mathbf{k}}\bigg(& \frac{1}{4}\Big|z_m(\mathbf{k})z_n(\mathbf{k}+\mathbf{e}_m) - z_n(\mathbf{k})z_m(\mathbf{k}+\mathbf{e}_n)\Big|^2 \\
& + \frac{1}{8}\Big(z_m(\mathbf{k})\bar{z}_m(\mathbf{k}) - \bar{z}_m(\mathbf{k}-\mathbf{e}_m)z_m(\mathbf{k}-\mathbf{e}_m)\Big)^2 \\
& + \eta(\mathbf{k})\Big(\bar{z}_m(\mathbf{k}+\mathbf{a}-\mathbf{e}_m)\psi_m(\mathbf{k}+\mathbf{a}-\mathbf{e}_m) - \psi_m(\mathbf{k}+\mathbf{a})\bar{z}_m(\mathbf{k}+\mathbf{a})\Big) \\
& - \frac{1}{2}\chi_{mn}(\mathbf{k})\Big(z_m(\mathbf{k})\psi_n(\mathbf{k}+\mathbf{e}_m) - \psi_n(\mathbf{k})z_m(\mathbf{k}+\mathbf{a}_n) \\
& \qquad\qquad - z_n(\mathbf{k})\psi_m(\mathbf{k}+\mathbf{e}_n) + \psi_m(\mathbf{k})z_n(\mathbf{k}+\mathbf{a}_m)\Big)\bigg). \tag{2.16}
\end{aligned}
$$

This is nothing but the action given in the link approach [16] with the identifications,[2]

$$z_m \equiv \sqrt{2}\mathcal{U}_{-m}, \quad \bar{z}_m \equiv \sqrt{2}\mathcal{U}_{+m}, \quad \eta \equiv i\rho, \quad \chi_{12} \equiv i\tilde{\rho}, \quad \psi_m \equiv \sqrt{2}\lambda_m, \quad (2.17)$$

where the right hand sides correspond to the notation used in [16]. The relations (2.15) among $\mathbf{e}_m$, $\mathbf{a}$, $\mathbf{a}_m$ and $\mathbf{a}_{12}$ are also as given in [16]. We see that they are nothing but the charge assignments for the fields, and in particular the shift variables $\mathbf{a}$, $\mathbf{a}_m$ and $\mathbf{a}_{12}$ are the $\mathrm{U}(1)$ charges of the fermions, a point of importance below. A related issue pertains to the three-dimensional structure discussed in [15] and which can be understood in terms of the maximal number of $\mathrm{U}(1)$ symmetries of the mother theory. In the following discussion, we concentrate for simplicity on the case of $\mathbf{a} = 0$. It is straightforward to extend the discussion to the general case.

We now turn to the question of preserved supersymmetries of the orbifolded theory. As discussed in [1], the orbifolded action is expected to be invariant only under the action of the scalar supercharge $\widehat{Q}$, a singlet under all $\mathrm{U}(1)$'s. This is in agreement with the naive expectation that only supersymmetries that do not generate space-time translations, even discrete ones, can be preserved in general. One can see this explicitly as follows. Consider the supersymmetry transformation (2.7) and the charge assignment of the fields. As we stressed above, the fermionic parameters $\hat{\kappa}$, $\hat{\kappa}_m$ and $\hat{\kappa}_{12}$ must be proportional to the unit matrix in order that (2.7) be a consistent set of transformations that leave the action invariant. After orbifolding this is simply impossible. If the corresponding transformations in the orbifolded theory should be meaningful at all, we are forced assign $\mathrm{U}(1)$ charges $\mathbf{0}$, $\mathbf{e}_m$ and $-\mathbf{e}_1 - \mathbf{e}_2$ to $\hat{\kappa}$, $\hat{\kappa}_m$ and $\hat{\kappa}_{12}$, respectively. In order that the transformation (2.7) be consistent with the orbifold projection, the $\hat{\kappa}_A$ must thus take the form

$$\hat{\kappa} = \kappa \mathbf{1}_{kN^2}, \quad \hat{\kappa}_m = \kappa_m V_{\mathbf{e}_m}, \quad \hat{\kappa}_{12} = \kappa_{12} V_{-\mathbf{e}_1 - \mathbf{e}_2}, \quad (2.18)$$

where κ, κ_m and κ_{12} are Grassmann parameters and $V_{\mathbf{q}}$ is defined as

$$V_{\mathbf{q}} \equiv \sum_{\mathbf{k}} \mathbf{1}_k \otimes E_{\mathbf{k}, \mathbf{k}+\mathbf{q}} . \quad (2.19)$$

This is the essential reason why the supersymmetries corresponding to $\widehat{Q}_{12}$ and $\widehat{Q}_m$ are broken after orbifolding. In fact, as emphasized above, the ordinary variation δS of the action (2.6) under (2.7) is zero only when the supersymmetry parameters are proportional to the unit matrix, using of course the usual Leibniz rule of variations,

$$\delta(FG) = (\delta F)G + F(\delta G) . \quad (2.20)$$

This conventional Leibniz rule for the variations of matrices in the mother theory leads to a modified rule for $\widehat{Q}_A = \{\widehat{Q}, \widehat{Q}_{12}, \widehat{Q}_m\}$,

$$\widehat{Q}_A(FG) = (\widehat{Q}_A F)G + (-1)^{|F|} V_A F V_A^{-1} (\widehat{Q}_A G), \quad (2.21)$$

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[2]We understand that this equivalence was known to the authors of ref. [16]. (N. Kawamoto, private communication)

where V_A expresses 1_{kN^2}, $V_{-\mathbf{e}_1-\mathbf{e}_2}$ and $V_{\mathbf{e}_m}$ corresponding to $\widehat{Q}$, $\widehat{Q}_{12}$ and $\widehat{Q}_m$, respectively. It is easy to show that the action S is not invariant under the transformations generated by $\widehat{Q}_{12}$ and $\widehat{Q}_m$ due to the modified Leibniz rule (2.21). Furthermore, with this modified rule the supersymmetry algebra (2.10) is not satisfied when acting on the multiplet of fields. Therefore, only the supercharge $\widehat{Q}$ which is associated with $\hat{\kappa}$ is preserved after the orbifold projection.

For the purpose of the discussion below, let us define supercharges that act on lattice fields, the field variables *after* orbifolding. Corresponding to the infinitesimal fermionic parameters (2.18), we see that the supercharges can be expressed as matrices as well:

$$\widehat{Q} \equiv Q1_{kN^2}, \quad \widehat{Q}_{12} \equiv Q_{12}V_{\mathbf{e}_1+\mathbf{e}_2}, \quad \widehat{Q}_m \equiv Q_mV_{-\mathbf{e}_m}. \tag{2.22}$$

This definition arises from the fact that the variations (2.7) corresponding to the supersymmetry transformations carry no U(1) charges. The two expressions $\{\widehat{Q}_A\} = \{\widehat{Q}, \widehat{Q}_{12}, \widehat{Q}_m\}$ and $\{Q_A\} = \{Q, Q_{12}, Q_m\}$ are completely equivalent after orbifolding, but the former act on the large matrices $z_m, \cdots$ and the latter act on the lattice fields $z_m(\mathbf{k}), \cdots$. Using $\{\kappa_A\}$ and $\{Q_A\}$, the supersymmetry transformation can be combined into

$$\delta\Phi = 2i\kappa Q\Phi - 2i\kappa_{12}Q_{12}\Phi + 2i\kappa_m Q_m\Phi . \tag{2.23}$$

In terms of the lattice fields it can be written as

$$\begin{aligned}
\delta z_m(\mathbf{k}) &= 2i\kappa\psi_m(\mathbf{k}) + 2i\kappa_m\eta(\mathbf{k}), \\
\delta\bar{z}_m(\mathbf{k}) &= -2i\kappa_{mn}\psi_n(\mathbf{k}-\mathbf{e}_n) - 2i\kappa_n\chi_{mn}(\mathbf{k}), \\
\delta\eta(\mathbf{k}) &= \frac{i}{2}\kappa\Big(z_m(\mathbf{k})\bar{z}_m(\mathbf{k}) - \bar{z}_m(\mathbf{k}-\mathbf{e}_m)z_m(\mathbf{k}-\mathbf{e}_m)\Big) \\
&\quad + i\kappa_{12}\Big(z_1(\mathbf{k}-\mathbf{e}_1-\mathbf{e}_2)z_2(\mathbf{k}-\mathbf{e}_2) - z_2(\mathbf{k}-\mathbf{e}_1-\mathbf{e}_2)z_1(\mathbf{k}-\mathbf{e}_1)\Big), \\
\delta\chi_{12}(\mathbf{k}) &= -i\kappa\Big(\bar{z}_1(\mathbf{k}+\mathbf{e}_2)\bar{z}_2(\mathbf{k}) - \bar{z}_2(\mathbf{k}+\mathbf{e}_1)\bar{z}_1(\mathbf{k})\Big) \\
&\quad - \frac{i}{2}\kappa_{12}\Big(z_m(\mathbf{k})\bar{z}_m(\mathbf{k}) - \bar{z}_m(\mathbf{k}-\mathbf{e}_m)z_m(\mathbf{k}-\mathbf{e}_m)\Big), \\
\delta\psi_m(\mathbf{k}) &= i\kappa_n\Big(z_m(\mathbf{k}+\mathbf{e}_n)\bar{z}_n(\mathbf{k}+\mathbf{e}_m) - \bar{z}_n(\mathbf{k})z_m(\mathbf{k}) \\
&\quad - \frac{1}{2}\delta_{mn}\Big(z_l(\mathbf{k})\bar{z}_l(\mathbf{k}) - \bar{z}_l(\mathbf{k}-\mathbf{e}_l)z_l(\mathbf{k}-\mathbf{e}_l)\Big)\Big).
\end{aligned} \tag{2.24}$$

From the charge assignment for the supercharges, we see that Q, Q_m and Q_{12} live on sites, links and diagonal links (or, equivalently, corners), respectively. Therefore, the actions of Q_m and Q_{12} change the geometrical structure of operators. For example, Q_m changes the link variable $z_m(\mathbf{k})$ into a site variable $\eta(\mathbf{k})$ as shown in the first line of (2.24).

We note that the operators Q_A obey a usual Leibniz rule,

$$Q_A(F(\mathbf{k})G(\mathbf{k}+\mathbf{e}_F)) = (Q_AF(\mathbf{k}))\,G(\mathbf{k}+\mathbf{e}_F) + (-1)^{|F|}F(\mathbf{k})\,(Q_AG(\mathbf{k}+\mathbf{e}_F)), \tag{2.25}$$

where $F(\mathbf{k})$ and $G(\mathbf{k})$ are lattice fields obtained from matrices F and G with U(1) charges $\mathbf{e}_F$ and $\mathbf{e}_G$, respectively. We stress that (2.25) is equivalent to (2.21).

One notices that the there is an ambiguity in the definition of (2.24). The transformation (2.24) is determined from (2.7) using the definition (2.23). However, there is no a priori principle to determine the positions of $\hat{\kappa}_A$ in (2.7) and the rule of transformation (2.24) depends on the positions of these fermionic parameters, since $\hat{\kappa}_A$ do not commute with other fields in general.

3. Equivalence between the orbifolding procedure and the link approach

In the previous section we have shown that the lattice action given by the link approach can be completely reproduced by the orbifolding procedure. However, as explicitly demonstrated above, only the supercharge with zero U(1) charges is preserved after orbifolding. The two actions being identical, this presents a puzzle in view of the arguments [16, 17] that in the link approach *all* supersymmetries are preserved. In this section, we show how also this claim can be understood in terms of the orbifolding procedure.

A first and interesting observation is that the transformations (2.24) coincide with those given of the link approach [16] under the identification (2.17). Nevertheless, we cannot identify Q_A with the supercharges in the link approach, s_A. The most important properties of s_A are (1) they satisfy a modified Leibniz rule when acting on lattice fields,

$$s_A(F(\mathbf{k})G(\mathbf{k} + \mathbf{e}_F)) = (s_A F(\mathbf{k})) \, G(\mathbf{k} + \mathbf{e}_F) + (-1)^{|F|} F(\mathbf{k} - \mathbf{e}_A) \, (s_A G(\mathbf{k} + \mathbf{e}_F)), \quad (3.1)$$

and (2) they satisfy the supersymmetry algebra corresponding to (2.10). However, the operators Q_A do not possess both of these properties. In fact, the Q_A's obey the usual Leibniz rule (2.25), and the only preserved part of the supersymmetry algebra is the one associated with nilpotency of the scalar charge Q, as mentioned in the previous section.

However, the operators Q_A turn out to satisfy the above two properties if we *impose* the usual Leibniz rule for $\widehat{Q}_A$. This is potentially confusing, but it corresponds to imposing

$$\widehat{Q}_A(FG) = \left(\widehat{Q}_A F\right) G + (-1)^{|F|} F \left(\widehat{Q}_A G\right), \quad (3.2)$$

instead of (2.21), without altering the transformations (2.7). In fact, if we impose (3.2) by hand, we derive the correspondingly modified Leibniz rule for Q_A,

$$Q_A(F(\mathbf{k})G(\mathbf{k} + \mathbf{e}_F)) = (Q_A F(\mathbf{k})) \, G(\mathbf{k} + \mathbf{e}_F) + (-1)^{|F|} F(\mathbf{k} - \mathbf{e}_A) \, (Q_A G(\mathbf{k} + \mathbf{e}_F)), \quad (3.3)$$

which coincides with (3.1). Moreover, it is straightforward to see that $\widehat{Q}_A$ satisfy the supersymmetry algebra (2.10) even after orbifolding if one imposes eq. (3.2). We conclude that the supercharges introduced in the link approach can be identified with the orbifolded supercharges of the mother theory (2.9) after demanding by hand the unusual Leibniz rule (3.2). We note that this argument is unchanged under an assignment of non-zero U(1) charges to η as in (2.16). So the equivalence holds in general.

Although the supercharges $\widehat{Q}_A$ (or Q_A) with the unusual Leibniz rule do not generate supersymmetries in any usual sense, the modified Leibniz rule in the orbifolded theory (3.3)

is actually consistent with gauge symmetry of the lattice theory. That is, the supersymmetry transformations (2.24) commute with gauge transformations. As an example, let us consider a supersymmetry transformation,

$$Q_{12}\bar{z}_1(\mathbf{k}) = \psi_2(\mathbf{k} - \mathbf{e}_2). \tag{3.4}$$

Since $\psi_2(\mathbf{k})$ is a link variable, the gauge transformation of the right hand side is

$$\psi_2(\mathbf{k} - \mathbf{e}_2) \rightarrow g^{-1}(\mathbf{k} - \mathbf{e}_2)\psi_2(\mathbf{k} - \mathbf{e}_2)g(\mathbf{k}). \tag{3.5}$$

On the other hand, let us first consider the gauge transformation of $\bar{z}_1(\mathbf{k})$,

$$\bar{z}_1(\mathbf{k}) \rightarrow g^{-1}(\mathbf{k} + \mathbf{e}_1)\bar{z}_1(\mathbf{k})g(\mathbf{k}). \tag{3.6}$$

Recalling the modified rule (3.3), we obtain

$$Q_{12}\big(g^{-1}(\mathbf{k} + \mathbf{e}_1)\bar{z}_1(\mathbf{k})g(\mathbf{k})\big) = g^{-1}(\mathbf{k} - \mathbf{e}_2)\psi_2(\mathbf{k} - \mathbf{e}_2)g(\mathbf{k}), \tag{3.7}$$

which is the same as (3.5). This illustrates the fact that the action of Q_A commutes with gauge transformation thanks to the modified Leibniz rule.

We close this section by pointing out that the question of possible additional symmetries of the orbifolded action appears even at the level of the mother theory, that is, in supersymmetric Yang-Mills matrix theory. As we have discussed, the supercharges of the link approach can be equivalently and compactly expressed as operators $\widehat{Q}_A$ that act on the large matrices in the orbifolded mother theory. From this point of view, all properties of the unusual supercharges $\widehat{Q}_A$ come from the matrix structure of the fermionic parameters $\hat{\kappa}_A$ as in eq. (2.18) and the modified Leibniz rule for $\widehat{Q}_A$ as in eq. (3.2). An important observation is that we can consider the transformation (2.9) with (2.18) and (3.2) in the framework of the mother theory without reference to the orbifold projection. Namely, we could imagine searching for additional symmetries of the mother theory (or, one higher level up, in the d-dimensional theory for which the mother theory is obtained by dimensional reduction) by allowing the non-trivial fermionic κ-parameters (2.18) and the modified Leibniz rule (3.2). The Leibniz rule of ordinary field variations is then also modified:

$$\delta^L(FG) = \big(\delta^L F\big)G + V_L^{-1}FV_L\big(\delta^L G\big). \tag{3.8}$$

The new examples based on the link approach correspond, at the level of the mother theory of matrices, precisely to this. This illustrates the problem (or challenge) in a quite transparent manner.

4. Conclusions

In this paper we have considered the relation between two lattice formulations of two-dimensional $\mathcal{N} = (2,2)$ supersymmetric gauge theory: the orbifolding procedure given in [2] and the link approach given in [16]. We have shown that the general action in the link approach can be obtained by the orbifolding procedure if one does not insist that one

fermionic field has zero $U(1)$ charges. We have written down the would-be supersymmetry transformations after orbifolding, and explicitly shown how they are broken by the projection. An interesting observation is that these transformations for the lattice fields coincide with those given in the link approach if one were allowed to introduce a matrix structure in the fermionic parameters $\hat{\kappa}_A$. They do not correspond to symmetries of the action in any usual sense. We have also shown that, by imposing a modified Leibniz rule for the original supercharges by hand, the supercharges after orbifolding can be identified with those of the link approach. As a result, the formulations based on the link approach and the orbifolding are equivalent. Any symmetries of the former are also symmetries of the latter, and vice versa. We have pointed out that the same issue can be discussed in the framework of supersymmetric Yang-Mills matrix theory.

Acknowledgments

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